IFT/2/96

hep-th/9601062

A Class of Exact Solutions of the Wheeler – De Witt Equation

Jerzy Kowalski-Glikman*
Institute for Theoretical Physics
University of Wrocław
Pl. Maxa Borna 9, PL–50-205 Wrocław, Poland
and
Krzysztof A. Meissner†
Institute of Theoretical Physics
Warsaw University
ul. Hoża 69, PL–00-681 Warsaw, Poland

January 1996

Abstract

After carefully regularizing the Wheeler – De Witt operator, which is the Hamiltonian operator of canonical quantum gravity, we find a class of exact solutions of the Wheeler – De Witt equation.

*e-mail address: jurekk@ift.uni.wroc.pl and jurekk@fuw.edu.pl

†Supported in part by the Polish KBN Grant; e-mail address: meissner@fuw.edu.pl

The enigma of quantum gravity is probably the most challenging problem of modern theoretical physics (for the recent reviews see [1]) It is for the difficulty of the problem that in spite of its importance, very little progress has been made so far. At the moment there are two major approaches to quantum gravity. On the one hand, it is being argued that gravity is a low energy approximation of some drastically different fundamental theory; the major examples of this kind of ideas are supergravities and, more recently, superstring theories. If one accepts such a proposal, "quantization" of classical general relativity does not make much sense.

There is however another approach in attempt to quantize the classical Einstein general theory of relativity. Nowadays the main line of attack lies in using canonical quantization technique. In recent years a great deal of excitement was risen by introduction of Ashtekar variables which were expected to simplify the technical problems of older canonical techniques. However, it seems that these hopes have not been fulfilled. Even though the use of the loop variables solves part of constraints automatically, it turned out that the remaining ones are as difficult to solve as the constraints of the traditional canonical approach, based on the Wheeler – De Witt equation (given the result of the present paper rather more difficult.) Moreover, due to the fact that existing solutions are expressed in terms of very exotic variables, their physical meaning is rather obscure. This is the reason why we choose to analyse again the original approach to quantum gravity which dates back to seminal papers of Wheeler and De Witt [2], [3]. It should be stressed however, that it seems likely that all these three approaches may reveal three different regimes of the same ultimate theory (see [1] for extensive discussion of this point).

The Wheeler – De Witt approach to quantization of general relativity is based on the following set of equations describing "the wavefunction of the universe":

$$\nabla_m \mathsf{p}^{mn}(x)\,\Psi(h_{mn}) = 0 \tag{1}$$

$$\mathsf{H}_{WDW}\Psi = \left(-\frac{1}{\mu}\frac{\mathcal{G}_{klmn}}{\sqrt{h}}\mathsf{p}^{kl}\mathsf{p}^{mn} + \mu\sqrt{h}\,(2\Lambda + R)\right)\Psi = 0, \tag{2}$$

where $\mu = (16\pi G)^{-1}$, G is the Newton's constant,

$$\mathcal{G}_{klmn} = \frac{1}{2}\left(h_{km}h_{ln} + h_{kn}h_{lm} - h_{kl}h_{mn}\right), \tag{3}$$

p^{kl} are momentum operators related to the three metric h_{mn},

$$\left[\mathsf{p}^{kl}(x),\, h_{mn}(y)\right] = i\delta^k_l\delta^l_n\delta^3(x-y), \tag{4}$$

R is the three dimensional curvature scalar, and Λ the cosmological constant. We assume also that the three-space is a compact smooth manifold without boundaries.

Equation (1) can be easily solved by assuming that the wavefunction is a function of integrals over space of scalar densities (local functionals). We will not use here any minisuperspace approximation so we take the lowest order local expressions that are diffeomorphism invariant. In what follows we will assume that

$$\Psi = \Psi(\mathcal{V}, \mathcal{R}), \tag{5}$$

where

$$\mathcal{V} = \int \sqrt{h(x)}d^3x, \qquad \mathcal{R} = \int \sqrt{h(x)}R(x)d^3x. \tag{6}$$

However, as it stands, equation (2) is meaningless. First the functional derivatives act at the same point, so acting on any local functional, the first term in (2) produces $\delta(0)$. This problem is resolved by regularization – in this paper we use the heat kernel to regularize divergent expressions. At this point one important remark is in order. The remarkable property of the metric representation is that the metric operator acts as multiplication. This means that we do not need to introduce any background metric for regularization purposes (and later work hard to make the final results background independent.) What we have for granted are eigenvalues of the metric operator on any state. Strictly speaking, in the heat kernel we should use metric operators, but these can be identified with their eigenvalues.

The second problem is related to the choice of ordering of the operator. The use of additional freedom given by the ordering parameter α will be essential in our procedure.

Thus we rewrite eq. (2) as

$$\Bigg(-\alpha\int d^3y\, \mathcal{G}_{klmn}(x)K(x,y;t)\frac{\delta}{\delta h_{kl}(x)}\frac{\delta}{\delta h_{mn}(y)} -$$
$$(1-\alpha)\int d^3y\, \frac{\delta\left(\mathcal{G}_{klmn}(x)K(x,y;t)\right)}{\delta h_{kl}(y)}\frac{\delta}{\delta h_{mn}(x)} + \mu^2\sqrt{h}(2\Lambda + R)\Bigg)\Psi = 0, \tag{7}$$

where $K(x,y;t)$ is a heat kernel satisfying the equation

$$\frac{\partial}{\partial t}K(x,y;t)=\nabla^2_{(x)}K(x,y;t)+\xi R(x)K(x,y;t) \tag{8}$$

with the initial condition

$$\lim_{t\to 0}K(x,y;t)=\frac{\delta^{(3)}(x-y)}{\sqrt{h(x)}}. \tag{9}$$

Let us note that in (7) we cannot take a more general ordering involving second functional derivatives of K, since these will again produce $\delta^{(3)}(0)$. Equation (8) can be solved pertubatively in powers of t. To the order which will be of interest in the present context, the solution reads [4]

$$\begin{aligned}K(x,y;t)=&\frac{\exp\left(\left(-\frac{1}{4t}h_{mn}(x)-\frac{1}{24}R_{mn}(x)\right)\Delta^m\Delta^n\right)}{(4\pi t)^{\frac{3}{2}}}*\\&\left(1+t\left(\xi-\frac{1}{6}\right)R(x)+t^2\left(\frac{1}{6}\left(\xi-\frac{1}{5}\right)\Box R(x)+\right.\right.\\&\left.\left.\frac{1}{2}\left(\xi-\frac{1}{6}\right)^2R^2(x)+\frac{1}{60}R_{mn}(x)R^{mn}(x)-\frac{1}{180}R^2(x)\right)+O(t^3)\right)\end{aligned} \tag{10}$$

where $\Delta^m=x^m-y^m$. In order to satisfy the Wheeler-De Witt equation we have to cancel $\Box R$ in the above expression so we choose $\xi=\frac{1}{5}$.

We will use below a shorthand notation

$$K_{klmn}(x,y;t):=\mathcal{G}_{klmn}(x)K(x,y;t) \tag{11}$$

We need one more technical result, namely the value of

$$\int d^3y\,\frac{\delta K_{klmn}(x,y;t)}{\delta h_{kl}(y)}. \tag{12}$$

This value cannot be derived by taking derivatives of terms in the expansion for K above (it involves terms of all orders in t), but can be found by solving perturbatively the equation for $\frac{\delta K}{\delta h_{mn}}$. After straightforward but tedious calculations one finds (to the leading order in t) that the expression (12) equals

$$\frac{1}{(4\pi)^{\frac{3}{2}}}\left(t^{-\frac{3}{2}}b_1h_{mn}(x)+t^{-\frac{1}{2}}(b_2R_{mn}(x)+b_3h_{mn}(x)R(x))+O(t^{\frac{1}{2}})\right).$$

where

$$b_1 = \frac{3}{4} + \frac{\xi}{3}, \quad b_2 = \frac{13\xi}{6}, \quad b_3 = \frac{\xi^2}{3} - \frac{5\xi}{12} - \frac{1}{8} \tag{13}$$

Now we are ready to present our main result. We assume that Ψ depends only on $\mathcal{V}$ and $\mathcal{R}$. Substituting this form to the Wheeler – De Witt equation (7) we find terms proportional to

$$\int d^3y\, K_{klmn}(x,y;t) \frac{\delta^2 \mathcal{V}}{\delta h_{kl}(x) \delta h_{mn}(y)} = -\frac{21}{8}\sqrt{h(x)} K(x,x;t) \tag{14}$$

$$\int d^3y\, K_{klmn}(x,y;t) \frac{\delta \mathcal{V}}{\delta h_{kl}(x)} \frac{\delta \mathcal{V}}{\delta h_{mn}(y)} = -\frac{3}{8}\sqrt{h(x)} \tag{15}$$

$$\int d^3y\, \frac{\delta K_{klmn}(x,y;t)}{\delta h_{kl}(y)} \frac{\delta \mathcal{V}}{\delta h_{mn}(x)} =$$
$$\frac{1}{(4\pi)^{\frac{3}{2}}} \left(t^{-\frac{3}{2}} \frac{3}{2} b_1 + t^{-\frac{1}{2}} (b_2 + 3b_3) R(x) + O(t^{\frac{1}{2}}) \right). \tag{16}$$

$$\int d^3y\, K_{klmn}(x,y;t) \frac{\delta^2 \mathcal{R}}{\delta h_{kl}(x) \delta h_{mn}(y)} =$$
$$\sqrt{h(x)} \left(-3 \frac{\partial K(x,x;t)}{\partial t} + (3\xi + 7/8) R(x) K(x,x;t) \right) \tag{17}$$

$$\int d^3y\, K_{klmn}(x,y;t) \frac{\delta \mathcal{R}}{\delta h_{kl}(x)} \frac{\delta \mathcal{R}}{\delta h_{mn}(y)} = \sqrt{h(x)} \left(R_{mn}(x) R^{mn}(x) - \frac{3}{8} R^2(x) \right) \tag{18}$$

$$\int d^3y\, K_{klmn}(x,y;t) \frac{\delta \mathcal{R}}{\delta h_{kl}(x)} \frac{\delta \mathcal{V}}{\delta h_{mn}(y)} = \sqrt{h(x)} \left(-\frac{1}{8} R(x) \right) \tag{19}$$

$$\int d^3y\, \frac{\delta K_{klmn}(x,y;t)}{\delta h_{kl}(y)} \frac{\delta \mathcal{R}}{\delta h_{mn}(x)} = \frac{1}{(4\pi)^{\frac{3}{2}}} \left(t^{-\frac{3}{2}} \frac{1}{2} b_1 R(x) + \right.$$
$$\left. t^{-\frac{1}{2}} \left((b_2 + b_3) R^2(x) - b_2 R_{mn}(x) R^{mn}(x) \right) + O(t^{\frac{1}{2}}) \right). \tag{20}$$

where b_i are given by (13).

Now we face the problem as to if renormalize the equation (i.e., replace the singular terms involving inverse powers of t) or to find a solution of

the equation and renormalize the so obtained wave function. Let us make this first choice. We use the method proposed by Mansfield [5] which, after multiplying by arbitrary function of t and analytical continuation results effectively in replacing the powers $t^{-\frac{p}{2}}$ by pth derivatives of arbitrary function $\phi(s)$ at $s = 0$, $(s = \sqrt{t})$ which we denote by $\phi^{(p)}$. These numbers are therefore renormalization constants.

Collecting all terms we have four equations (the coefficients multiplying $R^2(x)$, $R_{mn}R^{mn}$, $R(x)$ and 1 respectively):

$$\frac{3}{8}\frac{\partial^2\Psi}{\partial\mathcal{R}^2} + \frac{\partial\Psi}{\partial\mathcal{R}}\phi^{(1)}\left(\frac{3}{2}\alpha b_3 + \frac{1}{2}\alpha b_2 - \frac{19}{720}\right) = 0 \tag{21}$$

$$-\frac{\partial^2\Psi}{\partial\mathcal{R}^2} + \frac{\partial\Psi}{\partial\mathcal{R}}\phi^{(1)}\left(-\alpha b_2 - \frac{3}{80}\right) = 0 \tag{22}$$

$$\frac{1}{4}\frac{\partial^2\Psi}{\partial\mathcal{R}\partial\mathcal{V}} + \frac{\partial\Psi}{\partial\mathcal{R}}\phi^{(3)}\left(\frac{1}{2}\alpha b_1 - \frac{5}{8}\right) + \frac{\partial\Psi}{\partial\mathcal{V}}\phi^{(1)}\left(\frac{7}{80} + \frac{3}{2}\alpha b_3 + \frac{1}{2}\alpha b_2\right) + \mu^2\Psi = 0 \tag{23}$$

$$\frac{3}{8}\frac{\partial^2\Psi}{\partial\mathcal{V}^2} + \frac{\partial\Psi}{\partial\mathcal{V}}\phi^{(3)}\left(\frac{21}{8} + \frac{3}{2}\alpha b_1\right) + \frac{\partial\Psi}{\partial\mathcal{R}}\frac{3\phi^{(5)}}{4} + 2\Lambda\mu^2\Psi = 0 \tag{24}$$

These equations are linear differential equations with constant coefficients so the solutions are combinations of exponential functions of $\mathcal{V}$ and $\mathcal{R}$.

We will present an explicit solution when the wavefunction depends only on $\mathcal{V}$ (the general case is not difficult but the equations are rather lengthy). In this case only two of the equations survive and we substitute

$$\Psi(\mathcal{V}) = Ae^{\omega_1\mathcal{V}}, \tag{25}$$

Then we get from (23) and (24) that ω_1 satisfies

$$\frac{3}{8}\omega_1^2 + \frac{21}{8}\phi^{(3)}\omega_1 + 2\Lambda\mu^2 - \frac{3b_1\phi^{(3)}}{\phi^{(1)}(3b_3 + b_2)}\left(\frac{7}{80}\phi^{(1)}\omega_1 + \mu^2\right) = 0 \tag{26}$$

The dependence of the solution on $\mu^2 \sim G^{-2}$ can be written as

$$\omega_1 = -B \pm \sqrt{B^2 - C\mu^2} \tag{27}$$

where the constants B and C can be extracted from eq. (26). Depending on the values of the constants we get real or complex wave functions - the interpretation in these cases would be quite different.

In the most general case when the function depends on both $\mathcal{V}$ and $\mathcal{R}$ we get a relation between the cosmological constant and the Newton's constant expressed via the renormalization constants.

We conclude this paper with a number of comments:

1. It is natural to ask if one can find a solution depending on higher order local functionals like e.g. $S_2 = \int(bR^2 + cR_{mn}R^{mn})$. The procedure should be as follows. First, in order to avoid terms of order t^3 in the heat kernel expansion, we must choose b and c such as to avoid fourth order covariant derivatives in the second functional derivative of S_2 (otherwise we would have a large number of independent third order tensor structures rather impossible to match). This gives one relation between the coefficients. Then one should try to match terms with independent combinations of curvature and metric tensors up to the order three. It is unclear at the moment if this procedure leads to any nontrivial solution.

2. There are several directions of obvious generalizations of our result that are currently under investigations. First, one may consider inclusion of matter fields. The case of a scalar field and the closely related analysis of the cosmological constant problem will be the subject of the forthcomming paper. Second, it is of outmost importance for e.g., black holes physics to consider a theory based on a compact manifold with boundaries. This problem is, however, much more complicated.

3. It should be stressed that the wave function alone does not provide us with much insight into the physics of the solution. We certainly lack one important ingredient which is an inner product. In context of the solutions presented above, our goal will be to write down the path integral measure $\mu(h_{mn})dh_{mn}$ in the form
$$dh^{\perp}_{mn} d\mathcal{V} d\mathcal{R}\ J(h^{\perp}, \mathcal{V}, \mathcal{R}),$$
where $\delta h^{\perp}_{mn}$ are variations of the metric which leave $\mathcal{V}$ and $\mathcal{R}$ invariant:
$$h^{mn}\delta h^{\perp}_{mn} = 0, \quad R^{mn}\delta h^{\perp}_{mn} = 0$$
and then integrate over $h^{\perp}_{mn}$ to obtain
$$\mu(\mathcal{V}, \mathcal{R}) d\mathcal{V} d\mathcal{R}.$$

Only after succeeding this program we will be able to address questions concerning hermiticity of operators and their expectation values.

Alternatively, we may make use of the quantum potential approach to quantum gravity which provides us with the four dimensional dynamics related to a given wave function [6]. The major virtue of this approach is that most physical predictions concerning the evolution do not depend on the inner product.

4. Given the regularized WDW operator we can analyse the issue of anomalies. This problem has been investigated in [7], but we found ourselves in disagreement with the results of the papers. There are also additional, potentially anomalous terms resulting from the general ordering chosen in our approach. We are currently investigating this problem, and the result will be published soon.

5. One may wonder why such simple solutions were never found in spite of many papers on semiclassical expansion of the WDW equation. The reason is that in the semiclassical expansion, as a rule, one employs an expansion of the logarithm of the wave function in powers of G^{-2}. However, as our result shows, the exact solution is of the form e.g. $e^{\omega_1 \mathcal{V}}$ with ω_1 being a nonpolynomial function of G^{-2} (eq. (27)) therefore such an expansion is not finite (one should recall that the renormalization coefficients $\phi^{(n)}$ are dimensionful and therefore can mix with powers of G^{-2}).

References

[1] C. J. Isham in *Canonical Quantum Gravity: from Classical to Quantum*, Springer Verlag, Berlin, 1994; C. J. Isham in *Integrable Systems, Quantum Groups, and Quantum Field Theories*, L. A. Ibort and M. A. Rodriguez eds., Kluwer Academic Publishers, Dordrecht,1993; A. Ashtekar, *Non-Perturbative Canonical Quantum Gravity*, World Scientific, Singapore, 1991; C. J. Isham *Structural Issues in Quantum Gravity*, gr-qc 9510063.

[2] J.A. Wheeler in *Relativity, Groups and Topology*, C. De Witt and B.S. De Witt eds., Gordon and Breach, 1964; and in *Batelle Rencontres 1967*, C. De Witt and J.A. Wheeler eds., Benjamin 1967.

[3] Bryce S. De Witt, Phys. Rev. **160**, 1113, 1967.

[4] M. Lüscher, Ann. Phys. **142**, 359 (1982); Leonard Parker and David J. Toms, Phys. Rev. **D 31**, 953 (1985); F.A. Dilkes and D.G.C. McKeon, hep-th 9509005.

[5] Paul Mansfield, Nucl. Phys. **B 418**, 113 (1994).

[6] J. Kowalski-Glikman, *Quantum Potential and Quantum Gravity*, gr-qc 9511014, to appear in volume dedicated to Professor J. Lukierski, World Scientific, 1996.

[7] T. Horiguchi, K. Maeda, M. Sakamoto, Phys. Lett. **344 B**, 105 (1995); K. Maeda, M. Sakamoto, *Strong Coupling Quantum Gravity and Physics beyond the Planck Scale*, submitted to Phys. Rev. D.